mean $\overline{T}$; $\Theta(\tau) = T - T_0$, excess temperature of the probe relative to the initial temperature T_0; k_{RA}, amplification coefficient of RA; R_1, standard resistance in the bridge; R_2 and R_2^0, resistances of the potential divider at the RA input; V, voltage at the output of RA; D and L, diameter and length of the probe; $1/\alpha$, thermal resistance; $\gamma = 0.577215$, Euler's constant; and $\pi = 3.1415926$. Indices: 0, at the initial temperature T_0; s, referring to the sensor; c, to the calibrated sensor; and cz, to the contact zone.

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NUMERICAL SOLUTION OF THE INVERSE HEAT-CONDUCTION PROBLEM FOR DETERMINING THERMAL CONSTANTS

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UDC 536.24.01

We investigate the solution of the inverse heat-conduction problem for a cylinder, based on a series expansion of a thermal constant in powers of temperature, and the determination of the series coefficients by a direct-search method.

The majority of experimental methods of determination of the thermal constants of solid materials is based on the solution of the linear or nonlinear heat equation with some specific boundary conditions [1]. The use of these methods is brought about by the necessity of ensuring a stationary thermal regime, and monotonic or instantaneous heating to the required temperature which presents appreciable difficulties. In recent years it has been preferred to determine the thermal constants by the numerical solution of the inverse heat-conduction problem [2-7]. These methods do not as a rule, impose any restriction on the change of the boundary conditions. The physical parameter which appears in the heat equation is found in this case from the known boundary conditions and from the temperature at interior points.

In the present work we investigate a numerical solution of the inverse heat-conduction problem which can be immediately used for the experimental determination of the thermal conductivity or some other constant which appears in the heat equation. The solution is based on a series expansion of the required thermal constant in a series in powers of temperature and on the determination of the series coefficients by a specially derived method of direct search. We note that this method can be used for the solution of any one-dimensional heat-conduction problem with coefficients of interest, with any boundary conditions.

The problem of determination of a thermal constant from the experimentally measured values of temperature at two points of a sufficiently long, hollow, or dense cylinder can be represented by the following equations:

$$c\rho \frac{\partial t}{\partial \tau} = \frac{1}{r} \cdot \frac{\partial}{\partial r}\left(r\lambda \frac{\partial t}{\partial r}\right), \quad r_0 < r < R, \quad 0 < \tau < \tau_F, \tag{1}$$

$$t(r,\ 0) = \varphi(r), \tag{2}$$

$$t(R,\ \tau) = \psi(\tau), \tag{3}$$

$$\frac{\partial t(r_0,\ \tau)}{\partial r} = 0. \tag{4}$$

Institute of Technical Thermophysics, Academy of Sciences of the Ukrainian SSR, Kiev. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 33, No. 6, pp. 1058-1061, December, 1977. Original article submitted April 5, 1977.

These equations contain two unknown functions: temperature $t(r, \tau)$, and one of the thermal coefficients c, ρ, and λ. To complete the system of Eqs. (1)-(4) it is necessary to postulate the temperature at one point of the region, for example,

$$t(r_0, \tau) = \psi_1(\tau). \tag{5}$$

The function $\psi_1(\tau)$ is used in the search of the required coefficient. To be specific, we shall take this coefficient to be the thermal conductivity λ. The quantity λ is determined by successive approximations. If the function $\lambda_s(t)$ in the s-th approximation is known, we can find from the solution of Eqs. (1)-(4) the value of the discrete analog of the temperature function u_s in the s-th approximation. This is a correctly formulated direct problem and it can be solved, for example, by using the difference scheme. On the lattice

$$r_m = r_0 + mh;\ m = -1,\ 0,\ 1,\ \ldots,\ M;\ h = \frac{R - r_0}{M};$$

$$\tau_n = nl,\ n = 0,\ 1,\ \ldots,\ N,\ N = \frac{\tau_F}{l}$$

it has the form

$$u_m^{n+1} = u_m^n + \frac{l}{2h^2}\Big[(\lambda_m^n + \lambda_{m+1}^n)(u_{m+1}^n - u_m^n) - (\lambda_m^n + \lambda_{m-1}^n)(u_m^n - u_{m-1}^n)\Big] + \frac{l\lambda_m^n}{2hr_m}(u_{m+1}^n - u_{m-1}^n), \tag{6}$$

$$m = 0,\ 1,\ \ldots,\ M-1;$$

$$u_m^0 = \varphi(r_m),\ u_{-1}^{n+1} = u_1^{n+1},\ u_M^{n+1} = \psi(\tau_{n+1}). \tag{7}$$

To determine the variations of the function λ which decreases the difference between the given temperature $t(r_0, \tau_n) = \psi_1(\tau_n)$ and the temperature found from the solutions of Eqs. (6) and (7) at the points of the interval $0 < \tau < \tau_F$ we expand the function λ in a Taylor series in powers of u in the following fashion:

$$\lambda(v) = a_0 + a_1\left(1 - \frac{v}{\Delta t}\right) + a_2 v\left(1 - \frac{v}{\Delta t}\right) + a_3 v^2\left(1 - \frac{v}{\Delta t}\right) + \ldots, \tag{8}$$

where

$$v = u_0^n - t(r_0, 0),\ \Delta t = t(r_0, \tau_F) - t(r_0, 0).$$

The expansion (8) is convenient because the value of λ at the right end of the interval $[0, \tau_F]$ for $v = \Delta t$ depends only on the parameter a_0. The expansion of the thermal coefficient in powers of temperature reflects the nature of the problem since the analysis of experimental data shows that the temperature dependence of the thermal coefficients of various materials is usually reasonably well described by second- or third-order polynomials. In the numerical solution we shall keep a finite number of terms; this number will be denoted by J + 1. The coefficients of the series a_j, j = 0, 1, ..., J are determined from the condition that in J + 1 points of the interval $0 \le \tau \le \tau_F$ the functions $t(r_0, \tau_n)$ and u_0^n practically coincide. This condition will be taken in the form

$$\left|\frac{t(r_0\ \tau_n) - u_0^n}{t(r_0, \tau_n)}\right| < \delta, \tag{9}$$

where δ is usually taken equal to 10^{-9}. Each of the coefficients a_j, j = 0, 1, ..., J corresponds to a point τ_j of the interval $[0, \tau_F]$. The coefficient a_0 corresponds to $\tau = \tau_F$, and the remaining coefficients correspond to interior points of the interval $[0, \tau_F]$. The calculation of the coefficients a_j is carried out by successive approximations in cycles, starting from the first and ending with the approximation $a_{j(k)}$ when the coefficient and the corresponding point satisfy the expression (9). The value of the coefficient $a_{j(k)}$ of the previous cycle is the first approximation $a_{j(1)}$ of the next cycle. The next approximation is carried out when the coefficients $a_0, a_1, \ldots, a_{j-1}$ and the corresponding points satisfy (9). Consequently, each approximation for the coefficients a_j denotes the start of a new cycle of successive approximations for the coefficients $a_0, a_1, \ldots, a_{j-1}$. If $a_{j(k)}$ is the value of the coefficient a_j in the k-th approximation, the value of $a_{j(k+1)}$ is given by

$$a_{j(k+1)} = a_{j(k)} + [t(r_0, \tau^j) - u_{0(k)}^j]\frac{1}{W_{j(k)}}, \tag{10}$$

where $u_{0(k)}^j$ is the value of the discrete function calculated immediately before the (k+1)-th approximation for the coefficient a_j, and $W_{j(k)}$ is the absolute value of the rate of change of the function u_0^j with respect to the parameter a_j for the given cycle. It is determined after the second approximation $a_{j(2)}$ and in the subsequent approximations of the cycle it can be assumed to remain constant:

$$W_{j(k)} = \left| \frac{u^j_{0(k)} - u^j_{0(k-1)}}{a_{j(k)} - a_{j(k)}} \right|, \ k \geqslant 1. \tag{11}$$

For the determination of $a_{j(2)}$ the rate was taken from the previous cycle.

To check the above numerical method we carried out the following numerical experiment. We solved the direct heat-conduction problem for the thermal conductivity $\lambda = 1 + b_1 t + b_2 t^2 + b_3 t^3$, $b_\theta = 0\text{-}1$, and $\theta = 1$, 2, 3 with the initial values:

$$t(R, \tau) = b_4 + b_5\tau + b_6\tau^2, \ t(x, 0) = b_7,$$
$$c\rho = b_8, \ b_i = \text{const}, \ i = 1, 2, \ldots . \tag{12}$$

As a result, we determine the temperature values $t(r_0, \tau^j)$ for $\tau^j = \tau_F$, $^2/_3\,\tau_F$, and $^1/_3\,\tau_F$ which are used as the input data for the determination of the function λ by solving the inverse heat-conduction problem with the uniqueness conditions (12). The function λ was determined in the form of a series (8) in which we kept the first two or three terms.

The results of the calculations carried out in a sufficiently large range of b_i and τ_F give the following indications. To determine the thermal conductivity $\lambda(t(r_0, \tau))$ at $\tau = \tau_F$, $^2/_3\,\tau_F$, and $^1/_3\,\tau_F$ on a lattice with M = 10 we needed 5-7 min of computer time on the BÉSM-4M computer, indicating the simplicity and efficiency of the method.

The error λ decreases somewhat with shorter time interval τ_F. The value of τ_F is conveniently chosen from the expression

$$Fo_F = \frac{\lambda\tau_F}{c\rho(R - r_0)^2} = 0.05 - 0.1.$$

Keeping the third term in the series (8) which corresponds to the point $(r_0, \ ^2/_3\,\tau_F)$ improves the accuracy of the calculation only if the input data error, does not exceed some given value P*. The value of P* depends on b_1 and b_2. For example, for $b_1 = b_2 = b$, with b = 0.05, and $b_3 = 0$, we obtain P* = 0.05%; for b = 0.1 we have P* = 0.25%; for b = 0.3, P* = 1%, and for b = 0.5, P* = 2.5%. When solving the direct and inverse problems on the same lattice with no perturbation to the input data, the error in thermal conductivity does not exceed 0.01%. If the values $t(r_0, \tau_F)$ or $t(r_0, \ ^1/_3\,\tau_F)$ are given within the error P_t, the error in the thermal conductivity P_λ is $P_\lambda \cong 0.6\ P_t$, provided other data are known accurately. If the values $t(r_0, \tau)$ are known with the same accuracy (and there is some systematic error), we have $P_\lambda \cong 0.3\,P_t$. If the perturbation of the temperature $t(R, \tau)$ at the outer boundary of the region is of the form $\Delta = t(R, \tau) b_9 \sin(b_{10}\tau)$, $b_{10} = 1\text{-}100$, the error in the thermal conductivity is $P \cong b_9$. When all input data were known with error P, the error P_λ differed only little from P.

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